Boosting Taub-NUT to a BPS NUT-wave

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ABSTRACT: The boosted Taub-NUT metric with zero ADM mass is shown to possess a dual momentum in the direction of the boost giving credit to the existence of another 4-vector K_μ in linearized gravity associated to the NUT charge and dual to the usual P_μ. Taking the infinite boost limit we obtain a shock pp-wave with NUT charge. We show that the latter is the gravitational dual of the infinitely boosted Schwarzschild metric, also known as the Aichelburg-Sexl pp-wave. We review the fact that this new shock pp-wave is also a half-BPS solution of $\mathcal{N} = 1$ supergravity. It has a BPS bound equal to $K_t = |K_z|$.

KEYWORDS: Supersymmetry and Duality, Space-Time Symmetries, Black Holes.

JHEP01(2009)045

Contents

1. Introduction

Electric-magnetic duality was originally established as a symmetry of the Maxwell equations and revealed itself a powerful tool. As examples let us remind Dirac's realization [1] that the very existence of magnetic monopoles would imply the quantization of charges, or Montonen and Olive [2] who conjectured the presence of this symmetry inside non-abelian gauge theories which was later shown to hold as a strong-weak duality in $\mathcal{N} = 4$ super Yang-Mills [3].

Although Einstein's equations of General Relativity and the concept of electric-magnetic duality were already well-established, it was only in the 50s-60s that Taub [4] and Newman-Unti-Tamburino [5] discovered a solution, called the Taub-NUT solution, which possesses a mass but also another parameter called the NUT charge. It was soon realized [6] that this NUT charge could be understood as the gravitational magnetic dual of the ADM mass (see for example [7] and references therein). However, up to now, this duality, which acts as an Hodge operator on the Riemann tensor, has only been verified in linearized gravity and still resists attempts to be proven in the full non-linear theory [8, 9].

Motivated by the understanding of this important duality that seems to exist in General Relativity and the already-known BPS bound $M^2 + N^2 = Z^2$ [10], we were first interested in the supersymmetric properties of the charged Taub-NUT. We reviewed in our previous paper [11] how to obtain this BPS bound in $\mathcal{N} = 2$ supergravity, and derived the Killing spinors for this particular solution. We then showed the impossibility of including the NUT charge in the usual supersymmetry algebra and proposed a way of modifying the algebra such as to include it. This discussion was motivated by the projection we obtained on the Killing spinors and from Nester's construction. On the way, we also derived generalized expressions for the ADM and dual ADM 4-momenta. This construction made it clear that the vielbein formalism is certainly more appropriate to study the duality as it permits to express surface integrals in terms of regular spin connections (along the Misner string

direction). It was also noted that this completion of the $\mathcal{N} = 2$ supersymmetry algebra should already be present in $\mathcal{N} = 1$.

In this work we would like to give credit to the modification of the $\mathcal{N} = 2$ supersymmetry algebra presented in [11]. We provide evidence that the modification should also be present in $\mathcal{N} = 1$ and verify that it is indeed proportional to a 4-vector K_μ, where $K_0 = N$ for the Taub-NUT solution. In this letter, we thus consider the Taub-NUT solution with the ADM mass set to zero and with no electromagnetic charges. We shall refer to this solution as the "pure Taub-NUT". It is understood that this metric is solution of the vacuum Einstein equations and that it preserves no supersymmetries in $\mathcal{N} = 1$ supergravity as the BPS bound is $N = 0$.

The paper is organized as follows: In section 2, we apply our charge formulæ [11] to the boosted pure Taub-NUT and show that K_μ does transform as P_μ under boosts. In section 3, we study the limit of infinite boost using the method of Aichelburg-Sexl and show that we obtain a shock pp-wave. As expected from the result in [7] where it is checked that linearized pure Taub-NUT is dual to linearized Schwarzschild, we also recover the infinitely boosted Taub-NUT metric as the gravitational dual of the Aichelburg-Sexl pp-wave, or infinitely boosted Schwarzschild [12]. In section 4, we review the supersymmetric properties of pp-waves and show that the dual pp-wave preserves half of the supersymmetries and satisfies the BPS bound $K_0 = -K_3$. This is also checked calculating the charges of the dual pp-wave.

2. The boosted Taub-NUT solution

In this section, we wish to add credit to the existence of the 4-vector K_μ. This 4-vector was already presented in the derivation of the quantization condition in gravity in [7]. We also showed in [11] that the Nester construction or a formal definition of a gravitational dual analogue of the energy-momentum P_μ gives a unique expression for the 4-vector K_μ. The charge formulæ were however only applied to the static Taub-NUT. To show that this K_μ does transform as a 4-vector, we will boost the Taub-NUT solution and show that a momentum contribution appears in the boosted direction.

To do that, let us first recall the metric of the Taub-NUT solution which possesses a mass M and a NUT charge N:

$$ds^2 = -\frac{\lambda}{R^2}(dt + 2N\cos\theta d\phi)^2 + \frac{R^2}{\lambda}dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.1}$$

where we defined $\lambda = r^2 - N^2 - 2Mr$ and $R^2 = r^2 + N^2$. From now on, we will set $M = 0$ and discuss the pure Taub-NUT solution.

As we showed in [11], if we work in the vielbein formalism it is possible to derive an expression for P_μ and K_μ as surface integrals over the spin connection. The linearized vielbein is given by:

$$e^\mu = dx^\mu + \frac{1}{2}\eta^{\mu\nu}(h_{\nu\rho} + v_{\nu\rho})dx^\rho \tag{2.2}$$

where $h_{\nu\rho} = h_{\rho\nu}$ is the linearized metric, $v_{\nu\rho} = -v_{\rho\nu}$ is related to local Lorentz invariance and no difference is made between flat and curved indices as we are working in linearized

gravity around cartesian flat coordinates. The expressions for the charges in terms of the vielbein were found to be [11]:

$$P_0 = \frac{1}{16\pi} \oint (\partial_i h_{li} - \partial_l h_{ii} + \partial_i v_{il}) d\hat{\Sigma}_l, \tag{2.3}$$

$$P_k = \frac{1}{16\pi} \oint (\partial_0 h_{lk} - \partial_l h_{0k} + \delta_l^k \partial_i h_{0i} - \delta_l^k \partial_0 h_{ii} + \partial_k v_{0l} + \delta_l^k \partial_i v_{i0}) d\hat{\Sigma}_l, \tag{2.4}$$

$$K_0 = \frac{1}{16\pi} \oint \epsilon^{lij} (\partial_i h_{0j} + \partial_j v_{i0}) d\hat{\Sigma}_l, \tag{2.5}$$

$$K_k = \frac{1}{16\pi} \oint \epsilon^{lij} (\partial_i h_{kj} + \partial_j v_{ik}) d\hat{\Sigma}_l, \tag{2.6}$$

where $\epsilon^{123} = 1$. The only restriction for using these expressions is that the spin connection, using a particular linearized vielbein, has to be regular. For the Taub-NUT metric we obtained $K_0 = N$, $P_0 = M$ and $P_i = K_i = 0$ using a triangular vielbein.

One could easily argue that there always exists a gauge transformation such that the vielbein can be set in a symmetric gauge (and by this we mean $v_{\mu\nu} = 0$) which would reduce our formulæ for P_μ to the standard ADM ones. However, one should be aware that this is only valid (at the level of the calculation of charges) in the case where the gauge transformation is non-singular. In other words, one can use these formulæ in the symmetric gauge where $v_{\mu\nu} = 0$ only if the spin connection is regular along the Misner string direction. As stated in [11] we see that these expressions are generalized by saying that we should not fix the gauge in the symmetric gauge but rather in the "regular spin-connection" gauge.

A first natural test to certify the existence of the 4-vector K_μ is to show that the boosted Taub-NUT has $K_0 = \gamma N$ and a momentum in the direction of the boost equal to $\gamma \beta N$. As we are interested in calculating a surface integral at spatial infinity, we will directly work with the linearized pure Taub-NUT:

$$ds_{\text{Lin}}^2 = -d\bar{t}^2 - 4N \cos \bar{\theta} d\bar{\phi} d\bar{t} + d\bar{r}^2 + \bar{r}^2 (d\bar{\theta}^2 + \sin^2 \bar{\theta} d\bar{\phi}^2) \tag{2.7}$$

which can be written in cartesian coordinates as:

$$ds_{\text{Lin}}^2 = -d\bar{t}^2 - 4N \frac{\bar{z}}{\bar{r}} \frac{1}{\rho^2} (\bar{x} d\bar{y} - \bar{y} d\bar{x}) d\bar{t} + d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2 \tag{2.8}$$

where $\rho^2 = \bar{x}^2 + \bar{y}^2$.

If we now perform a boost in the z-direction:

$$\begin{aligned} \bar{t} &= \gamma(t - \beta z) & \bar{z} &= \gamma(z - \beta t) \\ \bar{x} &= x & \bar{y} &= y \end{aligned} \tag{2.9}$$

we get:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 - 4N \frac{\gamma^2 (z - \beta t)}{\bar{r} \rho^2} (dt - \beta dz)(x dy - y dx) \tag{2.10}$$

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One should be careful while treating the coordinate $\bar{r}$, as the large radius limit is really $r \equiv (x^2 + y^2 + z^2)^{1/2} \to \infty$, and thus:

$$
\begin{aligned}
\frac{1}{\bar{r}} &\equiv \left[x^2 + y^2 + \gamma^2 (z - \beta\, t)^2 \right]^{-1/2} \\
&= \frac{1}{r} \left[(\sin^2\theta + \gamma^2\cos^2\theta) + \gamma^2\beta^2(t^2/r^2) - 2\gamma^2\beta\cos\theta(t/r) \right]^{-1/2} \\
&\sim \frac{1}{rB} + O(1/r^2)
\end{aligned}
\tag{2.11}
$$

where we defined $B = \sqrt{sin^2\theta + \gamma^2 cos^2\theta}$.

Our choice for the vielbein is:

$$
\begin{aligned}
e^0 &= dt - 2N \frac{\gamma^2(z - \beta t)}{\bar{r}\rho^2} (y dx - x dy) \\
e^1 &= dx \\
e^2 &= dy \\
e^3 &= -2N \frac{\gamma^2\beta(z - \beta t)}{\bar{r}\rho^2} (y\, dx - x\, dy) + dz
\end{aligned}
\tag{2.12}
$$

where it can be checked that the spin connection is regular, in agreement with [11] because our choice is precisely the triangular vielbein $e^{\bar{m}}_{\mu}$ for the linearized static Taub-NUT metric transformed under the boost to $e^m_\mu = \Lambda^m_{\bar{n}}\, \Lambda_\mu{}^{\bar{\nu}} e^{\bar{n}}_{\bar{\nu}} = \delta^m_\mu + \frac{1}{2}\eta^{m\nu}(h_{\nu\mu} + v_{\nu\mu})$.

Looking at (2.10), the linear perturbations are:

$$
\begin{aligned}
h_{xz} &= -\beta h_{tx} = -2N\gamma^2\beta \frac{(z - \beta t)}{\bar{r}} \frac{y}{x^2 + y^2} \\
h_{yz} &= -\beta h_{ty} = \ \ 2N\gamma^2\beta \frac{(z - \beta t)}{\bar{r}} \frac{x}{x^2 + y^2}
\end{aligned}
\tag{2.13}
$$

And we can directly see in (2.12) that our vielbein gives us $v_{ta} = h_{ta}$ and $v_{za} = h_{za}$ for $a = x, y$.

We can now easily proceed to the calculation of K_0:

$$
\begin{aligned}
K_0 &= \frac{1}{16\pi} \oint \epsilon^{lij}(\partial_i h_{0j} + \partial_j v_{i0}) d\hat{\Sigma}_l = \frac{1}{8\pi} \oint \epsilon^{lij} \partial_i h_{0j} d\hat{\Sigma}_l \\
&= \frac{N}{4\pi}\gamma^2 \oint_S \frac{\sin\theta}{B^3} d\theta d\phi \\
&= \gamma N.
\end{aligned}
\tag{2.14}
$$

Note also that the time dependence in the integrand (2.14) is subleading and tends to zero when $r \to \infty$.

The calculation for K_z is readily the same and we find:

$$
K_z = -\frac{\beta}{8\pi} \oint \epsilon^{lij} \partial_i h_{0j} d\hat{\Sigma}_l = -\beta K_0 = -\gamma\beta N
\tag{2.15}
$$

while $K_x = K_y = 0$. Finally using (2.3) and (2.4), it is not difficult to show that $P_\mu = 0$ for the boosted Taub-NUT solution.[1]

We have thus shown that K_μ behaves as a 4-vector.

3. The pp-wave and its magnetic dual

In this section, we present two ways of obtaining the infinite boost of the Taub-NUT metric. The first derivation follows the steps of the method of Aichelburg and Sexl [12] who performed the infinite boost of the Schwarzschild metric. They obtained a (shock) pp-wave given by the expression:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 - 8\,p\,\ln(\sqrt{x^2 + y^2})\,\delta(t - z)\,(dt - dz)^2 \qquad (3.1)$$

This method was generalized in [13] and used, for example, for the infinite boost of the Reissner-Nordström black hole. This more general analysis was used in [14] for the infinite boost of the Kerr black holes where the Aichelburg-Sexl metric is shown to be recovered in a certain limit.

One important characteristic of (3.1) is that this metric is solution of the linearized but also of the full Einstein's equations. In fact, this kind of solutions was already well-known. The Aichelburg-Sexl metric belongs to the wider class of pp-waves, plane fronted waves with parallel rays, first introduced by Brinkmann in 1925 as metrics on Lorentzian manifolds:

$$ds^2 = H(u, x, y)du^2 - du\,dv + dx^2 + dy^2 \qquad (3.2)$$

where H is a smooth function. If moreover the function H is harmonic in x and y then it is a solution of the full Einstein's equations.

The specificity of the Aichelburg-Sexl solution is that the H function factorizes its u dependence in a delta function such that $H(u, x, y) = F(x, y)\delta(u)$ and $F(x, y)$ is a harmonic function. This shock pp-wave was also discussed in [15] and understood as the gravitational radiation of a particle travelling at the velocity of light measured by an observer at rest.

We now perform the infinite boost on the Taub-NUT solution. Like in the Schwarzschild case [12], we only need the linearized part of the metric. For the pure Taub-NUT we thus have:

$$ds^2 = -d\bar{t}^2 + d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2 + ds^2_{\text{def}} \qquad (3.3)$$

where $ds^2_{\text{def}} = -4N\cos\bar{\theta}d\bar{t}\,d\bar{\phi}$.

Here, for convenience, we will take the Misner string along the x direction (namely interchanging $\bar{x}$ and $\bar{z}$ in (2.8)) and boost along the z direction according to (2.9). We then

---

[1]For P_0 and P_z the integrands are zero while for P_x and P_y the integrands are non vanishing but the integrals (2.4) are zero.

find in the leading order in γ:

$$
\begin{aligned}
\bar{t} &\to \gamma\, u \\
\bar{z} &\to -\gamma\, u \\
\bar{r}^2 &\to \gamma^2\, u^2 + (x^2 + y^2) \\
\cos\bar{\theta} = \frac{\bar{x}}{\bar{r}} &\to x/\sqrt{\gamma^2\, u^2 + (x^2 + y^2)} \\
d\bar{\phi} &\to \frac{1}{\gamma}\, \frac{y\, du}{u^2 + \gamma^{-2}y^2}
\end{aligned}
\tag{3.4}
$$

where $\tan\bar{\phi} = \bar{y}/\bar{z}$, we defined $u = t - \beta z$, and by leading order we mean that $\bar{z} \to \gamma(z - \beta t) = -\gamma u + \gamma(1 - \beta)(z + t) \sim -\gamma u$. Note that we can also drop in $d\bar{\phi}$ the second term in $u\, dy$ as we will see that it is at $u = 0$, when the infinite boost limit is considered, that a contribution appears.

The deformed part of the metric becomes:

$$
ds^2_{\text{def}} = -4N\frac{x}{\sqrt{\gamma^2\, u^2 + (x^2 + y^2)}}\, \gamma du\, \frac{1}{\gamma}\, \frac{y\, du}{u^2 + \gamma^{-2}y^2}
\tag{3.5}
$$

In the limit of infinite boost, we take $\gamma \to \infty$ and $N \to 0$ while keeping $N\gamma = k$. This means we have:

$$
ds^2_{\text{def}} = -8k\, \lim_{\epsilon \to 0} \frac{1}{\epsilon}\, \frac{A\, du^2}{2\sqrt{(u/\epsilon)^2 + (1 + A^2)}((u/\epsilon)^2 + A^2)}
\tag{3.6}
$$

where we wrote $\epsilon = \gamma^{-1}x$ and $A = y/x$.

If we now take $\epsilon \to 0$ in the sense of the distributions using the fact that:

$$
\lim_{\epsilon \to 0} \frac{1}{\epsilon}f(z/\epsilon) = \delta(z)
\tag{3.7}
$$

for a function f such that $\int_{-\infty}^{+\infty} f(z)dz = 1$, we find:

$$
ds^2_{\text{def}} = -8k\, \arctan(1/A)\, \delta(u)\, du^2 = -8k\, \arctan(x/y)\, \delta(u)\, du^2
\tag{3.8}
$$

and the metric of the infinitely boosted pure Taub-NUT metric is:

$$
ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 - 8k\, \arctan(x/y)\, \delta(t - z)\, (dt - dz)^2
\tag{3.9}
$$

The metric (3.9) is obviously solution of the full non-linear Einstein equations because it is of the form (3.2) and $\arctan(x/y)$ is harmonic. Moreover, we show in the next section that $K_0 = -K_z = k$ as it should.

To confirm this limit we now describe an alternative derivation using the gravitational duality of (linearized) gravity [7]. We show that the infinite boost of the Taub-NUT metric is the gravitational dual of the Aichelburg-Sexl pp-wave. In fact it is enough to check that one metric has a Riemann tensor dual to the other. For simplicity, we will not take into account the Dirac delta function as it does not affect the following analysis.

The non-trivial fluctuations for the Aichelburg-Sexl pp-wave are:

$$h_{tt} = h_{zz} = -h_{tz} = -8\,p\,\ln(\sqrt{x^2 + y^2}) \tag{3.10}$$

The linearized Riemann tensor is defined as:

$$R_{\alpha\beta\gamma\delta} = 2\partial_{[\alpha}h_{\beta][\gamma,\delta]} \tag{3.11}$$

whose non-trivial components for the Aichelburg-Sexl metric are:

$$R_{tatb} = -\frac{1}{2}\,\partial_a\partial_b h_{tt} \quad R_{tazb} = -\frac{1}{2}\,\partial_a\partial_b h_{tz} \quad R_{zazb} = -\frac{1}{2}\,\partial_a\partial_b h_{zz} \tag{3.12}$$

for $a, b = x, y$ and where $R_{\alpha\beta\gamma\delta}$ is, as usual, antisymmetric in its first two and last two indices and symmetric under the exchange of the first and second pair of indices.

The infinitely boosted pure Taub-NUT has non-trivial fluctuations:

$$\tilde{h}_{tt} = \tilde{h}_{zz} = -\tilde{h}_{tz} = -8\,k\,\arctan(x/y) \tag{3.13}$$

where $\tilde{h}_{\mu\nu}$ refers to the dual metric. The non-trivial components of the Riemann tensor are thus the same as in (3.12) but with $h_{\mu\nu}$ replaced by $\tilde{h}_{\mu\nu}$.

It is then easy to check that the non-trivial components of the Riemann tensor for (3.9) are precisely the ones obtained from (3.12) by duality using:

$$\tilde{R}_{\alpha\beta\lambda\mu} = \frac{1}{2}\,\epsilon_{\alpha\beta\gamma\delta}\,R^{\gamma\delta}{}_{\lambda\mu} \tag{3.14}$$

with $\epsilon_{txyz} = 1$.

We have thus checked that the dual of the Aichelburg-Sexl pp-wave is the infinitely boosted Taub-NUT, which we will call the dual pp-wave or NUT-wave. The metric is another shock pp-wave but with a different harmonic function:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 - 8\,k\,\arctan(x/y)\,(dt - dz)^2 \tag{3.15}$$

Note that the harmonic function has a cut in the $x - y$ plane, remnant of the Misner string singularity. It is interesting to recall [11] that in the Killing spinor equation for Taub-NUT, the ADM mass and NUT charge appear in the combination $M - \gamma_5 N$ where $\gamma_5^2 = -1$, which is reminiscent of a complex structure. In the same way, for the pp-waves, we could construct a complex variable $\zeta = y + ix$ whose logarithm is $\ln\zeta = \ln\sqrt{x^2 + y^2} + i\arctan(x/y)$ and attribute the real part of this logarithm to the Aichelburg-Sexl metric and the imaginary part to the dual pp-wave. This last fact can be generalized to any solution (3.2), where $H(u, x, y) = F(x, y)\delta(u)$ and $F(x, y)$ is a harmonic function. The gravitational dual solution is characterized by $H(u, x, y) = \tilde{F}(x, y)\delta(u)$ where $\tilde{F}$ is the harmonic conjugate function of F (namely $\mathcal{F}(\zeta) = F + i\tilde{F}$ is an holomorphic function of ζ).[2]

___

[2]The holomorphic nature of $\mathcal{F}(\zeta)$ is reminiscent of the holomorphic nature of the complex Ernst potential for BPS solutions (see for instance section 3.4 of [16]).

4. Charges and supersymmetric properties of the dual pp-wave

In this section, we want to review the fact that the shock pp-wave is a supersymmetric solution of $\mathcal{N} = 1$ supergravity[3] and, as the BPS bound is $P_0 = -P_3$ for the Aichelburg-Sexl metric, we want to establish that the BPS bound is $K_0 = -K_3$ for our dual pp-wave. As a final check, we show that the charges for the dual pp-wave verify this BPS bound as it can be expected from the infinite boost of (2.14) and (2.15).

To use our formulae for the charges, we need a regular spin connection [11]. We will give arguments that the good choice of vielbein is the symmetric one. To do that, let us start with a pp-wave of the form:

$$
\begin{aligned}
ds^2 &= -dt^2 + dx^2 + dy^2 + dz^2 + F(dt - dz)^2 \\
&= -du(dv - Fdu) + dx^2 + dy^2
\end{aligned}
\tag{4.1}
$$

where $F = F(x, y)$ and where we defined light-cone coordinates $u = t - z$ and $v = t + z$. Note that we dropped again the delta function for simplicity.

An obvious vielbein choice in light-cone coordinates is:

$$
\begin{aligned}
e^- &= du & e^+ &= dv - Fdu \\
e^1 &= dx & e^2 &= dy
\end{aligned}
\tag{4.2}
$$

and the metric is $ds^2 = \eta_{ab}e^a\,e^b$ where the non-vanishing components are $\eta_{11} = \eta_{22} = 1$, $\eta_{+-} = \eta_{-+} = -1/2$.

Going back to cartesian coordinates, we obtain the symmetric vielbein:

$$
\begin{aligned}
e^0 &= \frac{1}{2}(e^+ + e^-) = dt - \frac{F}{2}(dt - dz) \\
e^1 &= dx \\
e^2 &= dy \\
e^3 &= \frac{1}{2}(e^+ - e^-) = dz - \frac{F}{2}(dt - dz)
\end{aligned}
\tag{4.3}
$$

where symmetricity is understood by the fact that $v_{\mu\nu} = -v_{\nu\mu} = 0$. The non-trivial components of the spin connection are:

$$
\omega_{0a} = -\omega_{3a} = \frac{1}{2}\partial_a F(x, y)(dt - dz)
\tag{4.4}
$$

where $F(x, y) = -8k \arctan(x/y)$ for the dual pp-wave. Even if in the case of our dual pp-wave the metric has a string singularity, one can see that the spin connection is "regular" in the $x - y$ plane . One could argue that a triangular vielbein with a regular linearized spin connection could also be used. However, it is important to note that our choice of vielbein is linear in the full theory. If one tries to construct such a triangular vielbein for example it would not be linear in the full theory and the operation of linearizing would then erase singularities in the spin connection such as $1/\sqrt{1 - F} \sim 1 + (1/2)F$ with F being the singular harmonic function. The calculation of charges would then fail.

---

[3]Note that all supersymmetric solutions of $\mathcal{N} = 1$ supergravity were classified in [17].

It can be easily seen that the pp-wave solution is a half-BPS solution of $\mathcal{N} = 1$ supergravity when looking at the Killing spinor equation (conventions are taken from [11]):

$$\delta\psi_\mu = \left[\partial_\mu + \frac{1}{4}\omega_\mu^{mn}\,\gamma_{mn}\right]\epsilon = 0 \tag{4.5}$$

This gives us the set of equations:

$$\begin{aligned}
\delta\psi_t &= \left[\partial_t - \frac{1}{4}\partial_a F(x,y)(\gamma_0 + \gamma_3)\gamma_a\right]\epsilon = 0 \\
\delta\psi_x &= \partial_x\epsilon = 0 \\
\delta\psi_y &= \partial_y\epsilon = 0 \\
\delta\psi_z &= \left[\partial_z + \frac{1}{4}\partial_a F(x,y)(\gamma_0 + \gamma_3)\gamma_a\right]\epsilon = 0
\end{aligned} \tag{4.6}$$

As the second and third equations show that ϵ does not depend on x and y, then the first and fourth equations imply the projection $(\gamma_0 + \gamma_3)\epsilon = 0$. This determines that the solution preserves half of the supersymmetries and has a constant Killing spinor. This projection corresponds to the BPS bound $K_0 = -K_3$ for our dual pp-wave.

As a final check, we calculate the charges for the dual pp-wave. For the Aichelburg-Sexl pp-wave, this was done in [18]. Here, in the symmetric vielbein, we have $v_{\mu\nu} = 0$ such that:

$$\begin{aligned}
K_0 &= \frac{1}{16\pi}\oint \epsilon^{lij}\partial_i h_{0j} d\hat{\Sigma}_l = -\frac{k}{2\pi}\oint \frac{1}{r}\delta(t-z)\,d\hat{\Sigma}_r \\
&= k\oint r\,\sin\theta\,\delta(t - r\cos\theta)\,d\theta = k\oint \delta(t - r\cos\theta)\,d(r\cos\theta) \\
&= k
\end{aligned} \tag{4.7}$$

Again the calculation for K_z is readily the same and gives $-k$. There is no contribution to P_μ.

5. Conclusions

In this letter, we have provided some more arguments to the fact that General Relativity, at least at the linear level, should include a 4-vector K_μ dual to the usual one P_μ.

Moreover, we showed that the infinite boost of Taub-NUT is a shock pp-wave and thus also a half-supersymmetric solution of $\mathcal{N} = 1$ supergravity. This provides more evidence that the NUT charge should be included in the $\mathcal{N} = 1$ supersymmetry algebra such as conjectured in [11]:

$$\{\mathcal{Q}, \mathcal{Q}'\} = \gamma^\mu C P_\mu + \gamma_5\gamma^\mu C K_\mu \tag{5.1}$$

where $\mathcal{Q}'$ is related to $\mathcal{Q}$ by a phase $\mathcal{Q}' = \mathcal{Q}\,e^{\alpha\gamma_5}$ with $\tan\alpha = K_0/P_0$. Indeed, the "modified" superalgebra (5.1) is consistent with the projection and the BPS bound derived in the previous section.

As a final word it would be interesting to see if the construction of more general dual supersymmetric solutions with NUT charge also provides modifications in their corresponding supersymmetric algebras such as in the super-AdS algebra. It would also be interesting to study further the appearance of the Lorentzian Taub-NUT charges in higher dimensional supersymmetry algebras such as the one characterizing M-theory [19].

Acknowledgments

References

[1] P.A.M. Dirac, *Quantised singularities in the electromagnetic field*, *Proc. Roy. Soc. Lond.* **A 133** (1931) 60.

[2] C. Montonen and D.I. Olive, *Magnetic monopoles as gauge particles?*, *Phys. Lett.* **B 72** (1977) 117.

[3] H. Osborn, *Topological charges for $N = 4$ supersymmetric gauge theories and monopoles of spin 1*, *Phys. Lett.* **B 83** (1979) 321.

[4] A.H. Taub, *Empty space-times admitting a three parameter group of motions*, *Ann. Math.* **53** (1951) 472.

[5] E. Newman, L. Tamubrino and T. Unti, *Empty space generalization of the Schwarzschild metric*, *J. Math. Phys.* **4** (1963) 915.

[6] M. Demianski and E.T. Newman, *A combined Kerr-NUT solution of the Einstein field equations*, *Bull. Acad. Pol. Sc.* **14** (1966) 653.

[7] C.W. Bunster, S. Cnockaert, M. Henneaux and R. Portugues, *Monopoles for gravitation and for higher spin fields*, *Phys. Rev.* **D 73** (2006) 105014 [hep-th/0601222].

[8] M. Henneaux and C. Teitelboim, *Duality in linearized gravity*, *Phys. Rev.* **D 71** (2005) 024018 [gr-qc/0408101].

[9] S. Deser and D. Seminara, *Free spin 2 duality invariance cannot be extended to GR*, *Phys. Rev.* **D 71** (2005) 081502 [hep-th/0503030].

[10] R. Kallosh, D. Kastor, T. Ortín and T. Torma, *Supersymmetry and stationary solutions in dilaton axion gravity*, *Phys. Rev.* **D 50** (1994) 6374 [hep-th/9406059].

[11] R. Argurio, F. Dehouck and L. Houart, *Supersymmetry and gravitational duality*, arXiv:0810.4999.

[12] P.C. Aichelburg and R.U. Sexl, *On the gravitational field of a massless particle*, *Gen. Rel. Grav.* **2** (1971) 303.

[13] C.O. Lousto and N.G. Sanchez, *The curved shock wave space-time of ultrarelativistic charged particles and their scattering*, *Int. J. Mod. Phys.* **A 5** (1990) 915.

[14] K. Hayashi and T. Samura, *Gravitational shock waves for Schwarzschild and Kerr black holes*, *Phys. Rev.* **D 50** (1994) 3666 [gr-qc/9404027].

[15] T. Dray and G. 't Hooft, *The gravitational shock wave of a massless particle*, *Nucl. Phys.* **B 253** (1985) 173.

[16] F. Englert, L. Houart, A. Kleinschmidt, H. Nicolai and N. Tabti, *An E_9 multiplet of BPS states*, *JHEP* **05** (2007) 065 [`hep-th/0703285`].

[17] K.p. Tod, *All metrics admitting supercovariantly constant spinors*, *Phys. Lett.* **B 121** (1983) 241.

[18] P.C. Aichelburg and H. Balasin, *ADM and Bondi four-momenta for the ultrarelativistic Schwarzschild black hole*, *Class. and Quant. Grav.* **15** (1998) 3841 [`gr-qc/9804007`].

[19] C.M. Hull, *Gravitational duality, branes and charges*, *Nucl. Phys.* **B 509** (1998) 216 [`hep-th/9705162`].